Data Dependencies in the Presence of Difference

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Outline

Introduction

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Motivation

Data Dependencies traditionally for quality of Schema:
schema design, integrity constraints, query optimization, etc.

Data Dependencies recently for quality of Data:
data cleaning, data repairing, record matching, etc.

Table: Example instance of Employee

<table>
<thead>
<tr>
<th>name</th>
<th>institute</th>
<th>title</th>
<th>salary</th>
<th>ssn</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>John Depp</td>
<td>Tech. Univ.</td>
<td>Professor</td>
<td>60</td>
</tr>
<tr>
<td>$t_2$</td>
<td>J. Depp</td>
<td>Technical Univ.</td>
<td>Professor</td>
<td>60</td>
</tr>
<tr>
<td>$t_3$</td>
<td>J.C. Depp</td>
<td>Tech. University</td>
<td>Prof.</td>
<td>3</td>
</tr>
<tr>
<td>$t_4$</td>
<td>R. Depp</td>
<td>Western Univ.</td>
<td>Lecturer</td>
<td>30</td>
</tr>
</tbody>
</table>
Motivation

Identification function in schema-oriented issues,

- in conventional dependencies, e.g., FDs
- title → salary
- $t_1$[title] : $Professor = t_2$[title] : $Professor$
- $t_1$[salary] : $60 = t_2$[salary] : $60$

Difference semantics in data-oriented practice,

- on numerical values or text values, e.g., similar or dissimilar.
- title : $Professor \approx Prof$
- salary: 60k v.s. 3k
Differential Dependencies: Syntax

We propose a novel type of dependencies

- *differential dependencies* (DDS)
- in the form of $\phi_L[X] \rightarrow \phi_R[Y]$
- $\phi_L[X]$ and $\phi_R[Y]$ are differential functions, which specify distance constraints on attributes $X$ and $Y$ of $R$, respectively.

Constraints on difference

- for any two tuples $(t_1, t_2)$ from an instance of $R$
- if their value differences (measured by certain distance metric) on attributes $X$ agree with the differential function $\phi_L[X]$, $(t_1, t_2) \simeq \phi_L[X]$  
- then their value differences on $Y$ should also agree with the differential function $\phi_R[Y]$, $(t_1, t_2) \simeq \phi_R[Y]$
Example

A \( \mathbb{DD} \) in a credit card transaction database can be

- \( \mathbb{DD}_1 \): \( [\text{cardno}(=0) \land \text{position}(\geq 60)] \rightarrow [\text{transtime}(\geq 20)] \)
- \( \text{cardno}(=0) \) states that two transactions have the same credit card no (the difference on attribute cardno is 0)
- \( \text{position}(\geq 60), \text{transtime}(\geq 20) \) are differential functions specified on attribute position, transtime, respectively

Constraints on difference

- If the distance of two transaction positions of a same cardno is \( \geq 60 \) km (e.g., two different cities)
- they are probably two transactions happening at different time
- the difference between transtime should be \( \geq 20 \) mins.

If two card transactions do not satisfy \( \mathbb{DD}_1 \), one of the transactions could be a fraud.
Example

A DD in a price database of a flight, in decision support systems

- \( \text{DD}_2 \quad \text{[date(\leq 7)]} \rightarrow \text{[price(\leq 100)]} \)

  states that the price difference of any two days in a week length should be less than 100 \\

Instead of a week length, another DD may specify

- \( \text{DD}_3 \quad \text{[date(> 7, \leq 30)]} \rightarrow \text{[price(> 100, \leq 900)]} \)

  the price difference constraint of two days not in a week length but in a month length

Both \( \text{DD}_2 \) and \( \text{DD}_3 \) specify

- on the same embedded attributes date \( \rightarrow \) price

- but with different constraint semantics, i.e., week and month.
Related Work

Conditional functional dependencies (CFDs)

- \((X \rightarrow A, t_p)\)
  - make the FDs, originally hold for the whole table, valid only for a set of tuples specified by the conditions
  - \(([\text{country, zip}] \rightarrow [\text{street}], < \text{Finland}, _ \parallel _ >)\)

Metric functional dependencies (MFDs)

- \(X \overset{\delta}{\rightarrow} A\)
  - similarity metrics in the right-hand-side, for violation detection
  - name \(\overset{2}{\rightarrow}\) address

Matching dependencies (MDs)

- \([X \approx] \rightarrow [A \Leftarrow]\)
  - “similar” semantics in the left-hand-side, for record matching
  - \([\text{name } \approx] \land [\text{addr } \approx] \rightarrow [\text{tel } \Leftarrow]\)
Comparison

CFDs introduce condition extension, which is still on identification semantics.

MFDs, MDs consider the “similar” semantics, on either determinant attributes \( X \) or dependent attributes \( Y \).

Our differential dependencies \( \text{DDS} \)

- \( \phi_L[X] \rightarrow \phi_R[Y] \)
- address more general difference constraints with various semantics
  - “similar” (e.g., \( \text{price}(\leq 100) \) in \( \text{DD}_2 \))
  - “dissimilar/different” (e.g., \( \text{transtime}(\geq 20) \) in \( \text{DD}_1 \)),
  - or even more complicated ones (e.g., \( \text{date}(> 7, \leq 30) \) in \( \text{DD}_3 \))
- allow setting difference constraints on both determinant attributes \( X \) and dependent attributes \( Y \)
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Example: Violation Detection

To find the tuples that violate dependencies

- according to $\text{DD}_2 \ [\text{date}(\leq 7)] \rightarrow [\text{price}(\leq 100)]$
- $t_3, t_4$ are detected as violations to $\text{DD}_2$

FDs cannot express such constraints on difference

- $t_3, t_4$ cannot be detected by a FD $\text{date} \rightarrow \text{price}$
- $t_1, t_2$ are detected as violations to FD by mistake

<table>
<thead>
<tr>
<th>Tuple</th>
<th>Date</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$</td>
<td>2010.06.01</td>
<td>1,000</td>
</tr>
<tr>
<td>$t_2$</td>
<td>2010.06.01</td>
<td>1,050</td>
</tr>
<tr>
<td>$t_3$</td>
<td>2010.08.02</td>
<td>2,000</td>
</tr>
<tr>
<td>$t_4$</td>
<td>2010.08.03</td>
<td>3,000</td>
</tr>
</tbody>
</table>
Evaluation: Violation Detection

DDs compared with FDs with identification functions
- differential functions in the right-hand-side $Y$
  - detect violations more accurately
  - the detection precision is higher than FDs
- differential functions in the left-hand-side $X$
  - address more tuples with violations
  - the detection recall by using DDs is higher than FDs

Figure: Violation detection accuracy
Example: Data Partition

To optimize data partition queries

- Integrity constraints (e.g., FDs or candidate keys) can be utilized to optimize the evaluation of queries
- known as the semantic query optimization

Consider a group-by query on distance conditions

```
SELECT * FROM Employee
GROUP BY institute(≤ 5) ∧ title(≤ 6)
```

- according to \([institute(\leq 5)] \rightarrow [institute(\leq 5) \wedge title(\leq 6)]\)
- rewrite the query by using institute(≤ 5) only

```
SELECT * FROM Employee
GROUP BY institute(≤ 5)
```
Evaluation: Data Partition

Using candidate differential key dependencies, CDK dependencies

- In x-axis, each element \( a/b \) corresponds to a pair of reduced/original differential functions for partitioning queries
  - \( a \) denotes the cardinality of CDK
  - \( b \) denotes the cardinality of original partition scheme
- the smaller the rate \( a/b \) is, the more the performance can be improved
Example: Record Linkage

To identify duplicate record, a.k.a. record matching, merge-purge

- use DDs as matching rules
  
  $\text{DD}_1 \quad [\text{name}(\leq 5) \land \text{institute}(\leq 7)] \rightarrow [\text{ssn}(= 0)]$
  
  $t_1, t_2$, whose name distance is $\leq 5$, and institute distance is $\leq 7$, probably denote the same employee with identical ssn

- another valid matching rule on same attributes
  
  $\text{DD}_2 \quad [\text{name}(\leq 3) \land \text{institute}(\leq 15)] \rightarrow [\text{ssn}(= 0)]$
  
  $t_2, t_3$ detected as duplicates by $\text{DD}_2$, not detected by $\text{DD}_1$

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Evaluation: Record Linkage

DDs compared MDs,

- MDs associate only one differential function on each attribute
- DDs can specify various differential functions on one attribute
- DDs address more matching rules
- recall of DDs is significantly higher
- DDs have comparable precision as MDs, both are valid matching rules

![Graph showing evaluation results for MDs and DDs in Restaurant data instances.](image-url)
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Differential Function: Intersection

The *intersection* of $\phi_1[Z]$ and $\phi_2[Z]$ on the same attributes $Z$ is

$$\phi_3[Z] = \phi_1[Z] \land \phi_2[Z]$$

- any $(t_1, t_2) \preceq \phi_1[Z]$ and $(t_1, t_2) \preceq \phi_2[Z]$, then $(t_1, t_2) \preceq \phi_3[Z]$
- any $(t_1, t_2) \not\preceq \phi_1[Z]$ or $(t_1, t_2) \not\preceq \phi_2[Z]$, then $(t_1, t_2) \not\preceq \phi_3[Z]$
- $[\text{name}(\leq 9)] \land [\text{name}(\leq 7)] = [\text{name}(\leq 7)]$

Apply intersection between $\phi_1[X]$ and $\phi_2[Y]$ on different attributes $X$ and $Y$

- Let $Z = X \cap Y$

$$\phi_1[X] \land \phi_2[Y] = (\phi_1[X \setminus Z] \land \phi_1[Z]) \land (\phi_2[Z] \land \phi_2[Y \setminus Z])$$

$$= \phi_1[X \setminus Z] \land (\phi_1[Z] \land \phi_2[Z]) \land \phi_2[Y \setminus Z].$$

- $[\text{name}(\leq 5) \land \text{address}(\leq 12)] \land [\text{address}(\leq 10)] = [\text{name}(\leq 5) \land \text{address}(\leq 10)]$
Differential Function: Subsumption

Intuitively, the semantics of “similar” subsumes identification
- any two values that are “identical” (with distance = 0)
- can always be interpreted as “similar” (with distance ≤ 9)

**Definition**

Let $\phi_1[Z]$ and $\phi_2[Z]$ be two differential functions on attributes $Z$
- If any tuple pair $(t_1, t_2) \simeq \phi_2[Z]$ always agree $(t_1, t_2) \simeq \phi_1[Z]$
- we say that $\phi_1[Z]$ *subsumes* $\phi_2[Z]$, written $\phi_1[Z] \succeq \phi_2[Z]$

For example
- $\phi_1[\text{name}] = [\text{name(≤ 9)}]$ subsumes $\phi_2[\text{name}] = [\text{name(≤ 7)}]$
  - denoted by $[\text{name(≤ 9)}] \succeq [\text{name(≤ 7)}]$
  - a distance value of name that agrees ≤ 7 will always agree ≤ 9
- $[\text{date(≤ 30)}] \succeq [\text{date(> 7, ≤ 30)}]$; $[\text{addr(≤ 9)}] \succeq [\text{addr(= 0)}]$
Differential Dependency

Consider an instance $I$ of relation $R$

- $(t_1, t_2) \sim \phi_L[X]$ denotes tuples $(t_1, t_2)$ having distance agreeing $\phi_L[X]$
- $I$ satisfies a DD, $I \models \phi_L[X] \rightarrow \phi_R[Y]$, if any two tuples $t_1$ and $t_2$ in $I$ having metric distances $(t_1, t_2) \sim \phi_L[X]$ must agree $(t_1, t_2) \sim \phi_R[Y]$
- $I$ satisfies a set $\Sigma$ of DDS, $I \models \Sigma$ if $I \models \phi_L[X] \rightarrow \phi_R[Y]$ for each $\phi_L[X] \rightarrow \phi_R[Y] \in \Sigma$.

Proposition

*For two differential functions $\phi_L[X]$ and $\phi_R[Y]$, if $Y \subseteq X$ and $\phi_R[Y] \succeq \phi_L[Y]$, then $\phi_L[X] \rightarrow \phi_R[Y]$.*

- a trivial DD, always holds
- $[\text{name}(\leq 5) \land \text{address}(\leq 10)] \rightarrow [\text{address}(\leq 12)]$
Logical Implication

Example

Consider two DDs,

\[ \text{DD}_4 \quad [\text{name}(\leq 7)] \rightarrow [\text{address}(\leq 1)], \]

\[ \text{DD}_5 \quad [\text{address}(\leq 5)] \rightarrow [\text{salary}(\leq 50)]. \]

- any two tuples \( t_1 \) and \( t_2 \) having name distance \( \leq 7 \),
- according to \( \text{DD}_4 \), their distance on address should be \( \leq 1 \),
- \((t_1, t_2)\) agree address\((\leq 5)\) as well.
- the salary distance of \( t_1 \) and \( t_2 \) should be \( \leq 50 \) according to \( \text{DD}_5 \)

We can imply another DD,

\[ \text{DD}_6 \quad [\text{name}(\leq 7)] \rightarrow [\text{salary}(\leq 50)]. \]
Implication Problem

Let $\Sigma_1$ and $\Sigma_2$ be two sets of DDs.

- $\Sigma_1$ logically implies $\Sigma_2$, $\Sigma_1 \models \Sigma_2$ if for all relation instance $I$, $I \models \Sigma_1$ implies $I \models \Sigma_2$

- $\Sigma_1$ and $\Sigma_2$ are equivalent, $\Sigma_1 \equiv \Sigma_2$ if $\Sigma_1 \models \Sigma_2$ and $\Sigma_2 \models \Sigma_1$

The implication problem

- given a consistent set $\Sigma$ of DDs and another DD $\phi_L[X] \rightarrow \phi_R[Y]$

- to decide whether $\Sigma$ can imply this DD, $\Sigma \models \phi_L[X] \rightarrow \phi_R[Y]$

- For example, $\{\text{DD}_4, \text{DD}_5\} \models \text{DD}_6$
Implication based-on Subsumption

Given a \( \text{DD} \) \( \phi_L[X] \rightarrow \phi_R[Y] \)

- \( \phi_1[Z] \rightarrow \phi_R[Y] \) can be implied, if \( X \subseteq Z, \phi_L[X] \succeq \phi_1[X] \)
- \( \phi_L[X] \rightarrow \phi_1[Z] \) can be implied, if \( Z \subseteq Y, \phi_1[Z] \succeq \phi_R[Z] \)

For example, consider a \( \text{DD} \) \([\text{name}(\leq 7)] \rightarrow [\text{address}(\leq 1)]\), it implies

- \([\text{name}(\leq 5)] \rightarrow [\text{address}(\leq 1)]\)
- \([\text{name}(\leq 7)] \rightarrow [\text{address}(\leq 2)]\)
Differential Key

Key: $t_1[R] = t_2[R]$ according to $t_1[K] = t_2[K]$ on a key $K \subseteq R$

A differential key $\phi_2[K]$ relative to $\phi_1[R]$

- is a differential function that can determine $\phi_1[R]$
- a differential key dependency $\phi_2[K] \rightarrow \phi_1[R]$ with $K \subseteq R$ and $\phi_2[K] \succeq \phi_1[K]$

For example,

- $[\text{position}(\geq 20)]$ is a differential key relative to $[\text{position}(\geq 20) \land \text{area}(\geq 5)]$
- according to the following differential key dependency, $[\text{position}(\geq 20)] \rightarrow [\text{position}(\geq 20) \land \text{area}(\geq 5)]$
Candidate Differential Key

A naïve key relative to $\phi_1[R]$ is $\phi_1[R]$ itself

A candidate differential key (CDK) $\phi_c[K]$ is

- an irreducible differential key relative to $\phi_1[R]$,
- there does not exist any $\phi_2[L]$ such that $L \subseteq K$, $\phi_2[L] \geq \phi_c[L]$ and $\phi_2[L] \rightarrow \phi_1[R]$.

A CDK

- not only has a minimal cardinality as candidate keys on FDs,
- but also should be the one not subsumed by others.

CDKs are useful in applications like data partition
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Discovery Problem

Discovery from data

- given a relation instance $I$
- discover candidate differential keys and minimal cover of differential dependencies that hold in $I$

The hardness

- a minimal cover of $FD$s, that hold in a relation instance $I$, can be exponentially large in the number of attributes
- $FD$s are considered as special cases of $DD$s where all the differential constraints are set to $= 0$
- $DD$s subsume $FD$s, could be exponentially large as well
Negative Pruning

Motivation: pruning candidates of DDSs, in order to avoid evaluating all possible $\phi_L[X] \rightarrow \phi_R[Y]$ in $I$

Lemma

For any $\phi_1[V], \phi_2[Z]$ having $V \subseteq Z, \phi_1[V] \succeq \phi_2[V]$, if $I \not\models \phi_2[Z] \rightarrow \phi_R[Y]$, then $I \not\models \phi_1[V] \rightarrow \phi_R[Y]$

Example: if $[\text{name}(\leq 5)] \rightarrow \phi_R[Y]$ not hold in $I$, then $[\text{name}(\leq 7)] \rightarrow \phi_R[Y]$ not hold either without evaluation in $I$

Worst case: all the candidates hold in the given instance $I$
Positive Pruning

**Lemma**

*For any* $\phi_1[V], \phi_2[W]$ *having* $W \subseteq V, \phi_2[W] \succeq \phi_1[W]$, *if* $I \models \phi_2[W] \rightarrow \phi_R[Y]$, *then* $I \models \phi_1[V] \rightarrow \phi_R[Y]$.

**Example:** if $[\text{name}(\leq 7)] \rightarrow \phi_R[Y]$ holds in $I$, then $[\text{name}(\leq 5)] \rightarrow \phi_R[Y]$ must hold without evaluation in $I$.

**Worst case:** all the candidates do not hold in the given instance $I$.

Hybrid approach with both positive and negative pruning, used by turns.
Instance Exclusion

**Motivation**: avoiding evaluating the entire $I$.

- one differential function subsumes another
- the set of tuples agreeing on the former one should be a super set of the latter one

Considers all the pairs of tuples in $I$.

\[
D(I) = \{ (t_i, t_j) \mid \forall t_i, t_j \in I \}.
\]

Given any DD $\phi_L[X] \rightarrow \phi_R[Y]$, we define $D(I, \phi_L[X], \neg \phi_R[Y]) =$

\[
\{(t_i, t_j) \in D(I) \mid (t_i, t_j) \bowtie \phi_L[X], (t_i, t_j) \not\bowtie \phi_R[Y] \},
\]

that is, the tuple pairs agreeing $\phi_L[X]$ but not agreeing $\phi_R[Y]$. 
Instance Exclusion

Lemma

An instance $I$ satisfies a DD, $I \models \phi_L[X] \rightarrow \phi_R[Y]$, iff $D(I, \phi_L[X], \neg \phi_R[Y]) = \emptyset$.

During the discovery, for a candidate $\phi_L[X] \rightarrow \phi_R[Y]$, have to evaluate whether $D(I, \phi_L[X], \neg \phi_R[Y]) = \emptyset$. 
Instance Exclusion

Lemma

For any $\phi_1[V], \phi_2[W]$ having $W \subseteq V, \phi_2[W] \succeq \phi_1[W]$, we have $D(I, \phi_1[V], \neg \phi_R[Y]) \subseteq D(I, \phi_2[W], \neg \phi_R[Y])$.

Suppose that a current $D(I, \phi_2[W], \neg \phi_R[Y]) \neq \emptyset$

- instead of considering the entire $D(I)$
- use $D(I, \phi_2[W], \neg \phi_R[Y])$ to compute $D(I, \phi_1[V], \neg \phi_R[Y])$
Experiments

Evaluate the time performance of discovery approaches

- scale well with the increase of tuples in an instance \( I \)
- \( O(n^2) \) with respect to the number of tuples \( n \) in the instance \( I \)
- instance exclusion performs well

Figure: DDs discovery performance on various instance \( I \)
Experiments

- discovery cost increases exponentially in the number of attributes in a schema
- can achieve several orders of magnitude improvement compared with brute-force one

Figure: DDs discovery performance on various schema $R$
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Conclusions

We propose a novel class of dependencies, differential dependencies (DDs), which specify constraints on distance.

Theory

- formal definitions of DDs and differential keys
- subsumption order relation of differential functions
- reasoning about DDs
  - consistency of DDs, NP-complete
  - implication of DDs, co-NP-complete
  - closure of a differential function
  - a sound and complete inference system, proof
  - minimal cover for DDs

Practice

- discovery of DDs and differential keys from data.
- application of DDs and differential keys.
Future Work

Approximate differential dependencies

- “almost” hold in a data instance
- evaluation measure, efficient computation
  - Implication of approximate differential dependencies
  - Hardness analysis of computing error measure
  - Approximation algorithms computing error measure
  - Experiments of approximation validation

Further extensions

- data repairing with DDs
- conditioning DDs in a subset of tuples
- integrity rules in dataspaces
Data Dependencies in the Presence of Difference

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The closure of $\phi_L[X]$ under $\Sigma$, $(\phi_L[X])^+$
- is also a differential function
- the intersection of the set of differential functions that can be determined by $\phi_L[X]$ according to DDS in $\Sigma$

$$ (\phi_L[X])^+ = \bigwedge \{ \phi_R[Y] \mid \Sigma \models \phi_L[X] \rightarrow \phi_R[Y] \} $$

- the closure of $[\text{name}(\leq 7)]$ under $\{\text{DD}_4, \text{DD}_5\}$ is $[\text{name}(\leq 7) \land \text{address}(\leq 1) \land \text{salary}(\leq 50)]$

It is natural that $\phi_L[X] \rightarrow (\phi_L[X])^+$.
Closure

To imply a DD is essentially to compute the corresponding closure \((\phi_L[X])^+\) of \(\phi_L[X]\)

**Lemma**

Let \(\Sigma\) be a set of DDs and \(\phi_1[Z] = (\phi_L[X])^+\) be the closure of \(\phi_L[X]\) with respect to \(\Sigma\).

- Consider a DD \(\phi_L[X] \rightarrow \phi_R[Y]\),
- \(\Sigma \vdash \phi_L[X] \rightarrow \phi_R[Y]\) iff \(Y \subseteq Z\) and \(\phi_R[Y] \supseteq \phi_1[Y]\).

For example,

- [salary(\(\leq 50\))] subsumes the projection on salary of the closure of [name(\(\leq 7\))] under \(\{DD_4, DD_5\}\)
- it implies \(DD_6\) [name(\(\leq 7\))] \(\rightarrow\) [salary(\(\leq 50\))]
Inference System

A1. If $Y \subseteq X$ and $\phi_L[Y] = \phi_R[Y]$, then $\Sigma \vdash \phi_L[X] \rightarrow \phi_R[Y]$.

A2. If $\Sigma \vdash \phi_L[X] \rightarrow \phi_R[Y]$, then
$$\Sigma \vdash \phi_L[X] \land \phi_1[Z] \rightarrow \phi_R[Y] \land \phi_1[Z].$$

A3. If $\Sigma \vdash \phi_L[X] \rightarrow \phi_1[Z]$, $\phi_1[Z] \preceq \phi_2[Z]$ and
$$\Sigma \vdash \phi_2[Z] \rightarrow \phi_R[Y],$$
then $\Sigma \vdash \phi_L[X] \rightarrow \phi_R[Y]$.

A4. If $\Sigma \vdash \phi_L[X] \land \phi_i[B] \rightarrow \phi_R[Y], 1 \leq i \leq k$, and
$$(\Sigma, \phi_1[B] \land \cdots \land \phi_k[B])$$ is inconsistent, then
$$\Sigma \vdash \phi_L[X] \rightarrow \phi_R[Y].$$

Theorem

The set $\mathcal{I}$ of inference rules is

- **(sound)**, if $\Sigma \vdash \phi_L[X] \rightarrow \phi_R[Y]$ then $\Sigma \models \phi_L[X] \rightarrow \phi_R[Y]$,
- **(complete)**, if $\Sigma \models \phi_L[X] \rightarrow \phi_R[Y]$ then $\Sigma \vdash \phi_L[X] \rightarrow \phi_R[Y]$,

for logical implication of DDS.
Example: Inference

Example

We consider a set $\Sigma$ of DDs as follows:

$$
\begin{align*}
\text{DD}_7 & \quad [d(\geq 1, \leq 7) \land p(<10)] \rightarrow [a(\leq 150)], \\
\text{DD}_8 & \quad [p(\geq 10)] \rightarrow [a(\leq 100)].
\end{align*}
$$

Let $\text{DD}_9$ be another DD

$$
\text{DD}_9 \quad [d(\geq 1, \leq 7)] \rightarrow [a(\leq 150)].
$$

We show that $\Sigma \vdash_{I} \text{DD}_9$ can be proved by the following steps.

1. $[d(\geq 1, \leq 7) \land p(\geq 10)] \rightarrow [d(\geq 1, \leq 7) \land a(\leq 100)]$ by $A2$, $\text{DD}_8$
2. $[d(\geq 1, \leq 7) \land a(\leq 150)] \rightarrow [a(\leq 150)]$ by $A1$
3. $[d(\geq 1, \leq 7) \land p(\geq 10)] \rightarrow [a(\leq 150)]$ by $A3$, 1. 2.
4. $[d(\geq 1, \leq 7)] \rightarrow [a(\leq 150)]$ by $A4$, 3. $\text{DD}_7$
Minimal Cover

A minimal cover $\Sigma_c$ for $\Sigma$ is a set of DDs such that $\Sigma_c$

- is logically equivalent to $\Sigma$, i.e., $\Sigma_c \equiv \Sigma$
- is minimal according to the following properties:

C1. (left-reduced), for any $\phi_L[X] \rightarrow \phi_R[Y] \in \Sigma_c$, there does not exist any $\phi_1[W]$ such that $W \subseteq X$, $\phi_1[W] \succeq \phi_L[W]$ and $\Sigma_c \models \phi_1[W] \rightarrow \phi_R[Y]$.

C2. (right-subsumed), for any $\phi_L[X] \rightarrow \phi_R[Y] \in \Sigma_c$, there does not exist any $\phi_1[W]$ such that $Y \subseteq W$, $\phi_1[Y] \preceq \phi_R[Y]$ and $\Sigma_c \models \phi_L[X] \rightarrow \phi_1[W]$.

C3. (non-redundant), there does not exist a cover $\Sigma'$ of $\Sigma$ such that $\Sigma' \subset \Sigma_c$. 
Example: Minimal Cover

Consider $\Sigma = \{\text{DD}_4, \text{DD}_5, \text{DD}_6\}$ in Example 2
- a minimal cover can be $\Sigma_c = \{\text{DD}_4, \text{DD}_5\}$
- $\Sigma_c$ can imply $\text{DD}_6$
- by removing $\text{DD}_4$ or $\text{DD}_5$ from $\Sigma_c$, it is no longer a cover of $\Sigma$

Consider $\Sigma = \{\text{DD}_7, \text{DD}_8, \text{DD}_9\}$ in Example 9
- a minimal cover can be $\Sigma_c = \{\text{DD}_8, \text{DD}_9\}$
- $\Sigma' = \{\text{DD}_7, \text{DD}_8\}$ is not a minimal cover
- since $\text{DD}_7$ is not left-reduced and can be implied by $\text{DD}_9$ by augmentation rule A2.